Solutions

MAT 342 Applied Complex Analysis
Midterm I, March 6, 2013

Name:____

I.D.:____

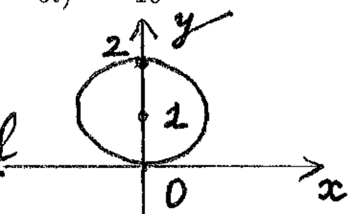
No calculators, computers, iPads or cellphones can be used on this test. Answer each question in the space provided and on the reverse side of the sheets. Show your work: no credit will be given for unjustified answers. You will find your test score on the last page of the exam.

1. (a) (15 points) Sketch the following sets, and determine which are open or closed, and which are bounded.

(i) $|\bar{z}+i| = 1$, (ii) $|z+1-i| < 3$, (iii) $\mathrm{Im}(\bar{z}-5i) = -10$

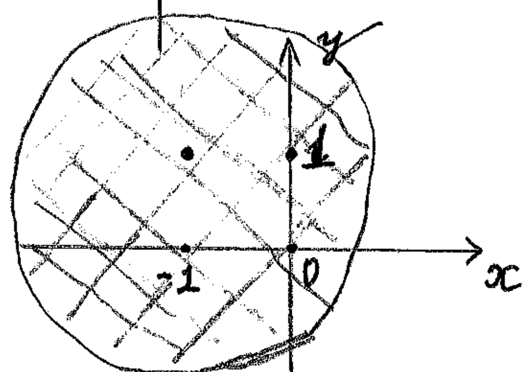
(i) $|\bar{z}+i| = |\overline{\bar{z}+i}| = |z-i| = 1$

- circle w/ center at i & radius 1 – closed & bounded



(ii) $|z-(i-1)| < 3$

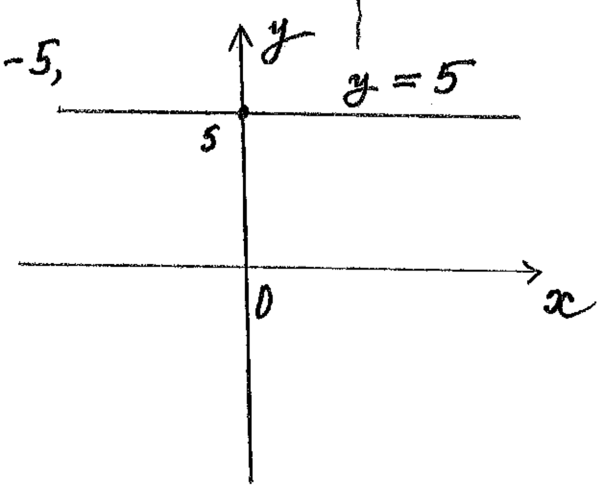
- interior of the disk w/ center at $i-1$ & radius 3
- open & bounded



(iii) $-10 = \mathrm{Im}(\bar{z}-5i) = -\mathrm{Im}\, z - 5$,

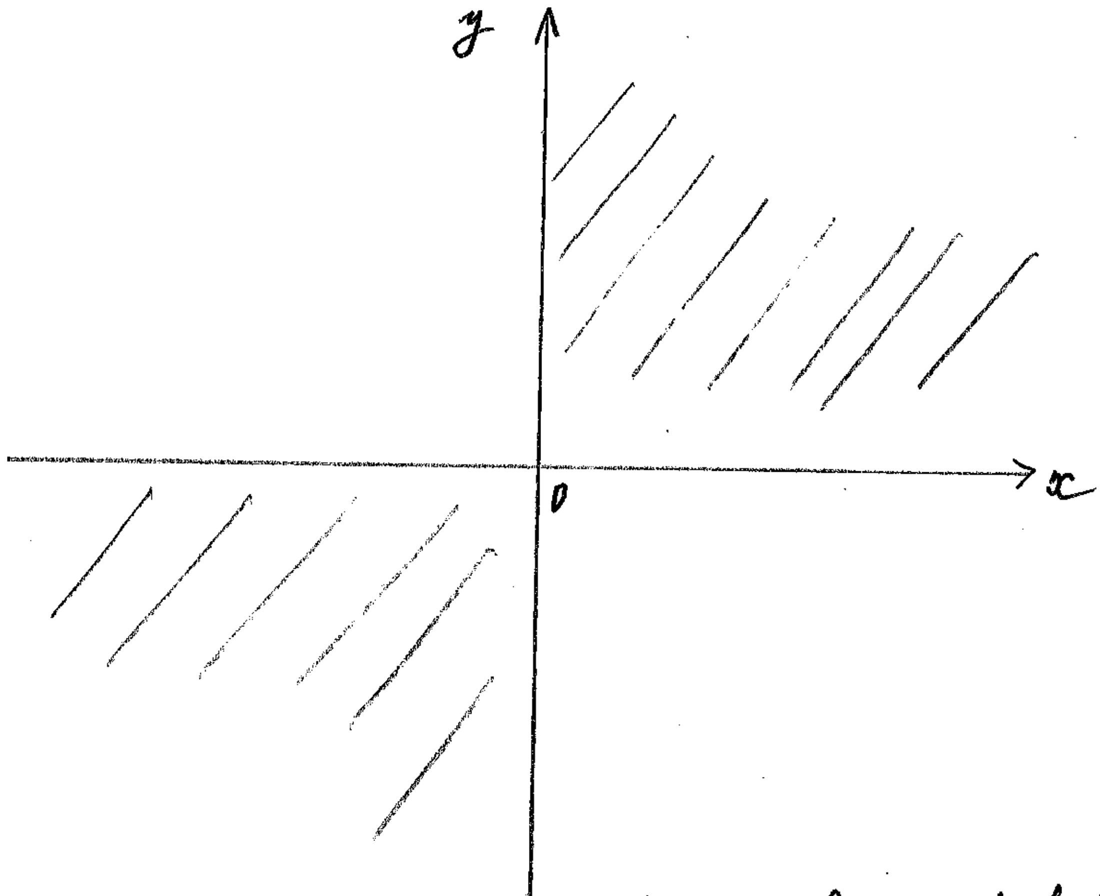
$\mathrm{Im}\, z = 5$

closed & unbounded



(b) (5 points) Sketch the set $\mathrm{Im}(z^2) > 0$ and describe its boundary and the closure.

$z = x + iy$, $\mathrm{Im}(z^2) = \mathrm{Im}(x^2 - y^2 + 2ixy)$
$= 2xy$
$xy > 0 \iff x > 0, y > 0$ & $x < 0, y < 0$



The set $\mathrm{Im}(z^2) > 0$ = interior of the 1st & 3rd quadrants of xy-plane (without the x & y axes)
Its boundary = x & y axes (lines $y=0$ & $x=0$)
Its closure = closed 1st & 3rd quadrants (with x & y axes)

2. (a) (10 points) Find all fourth roots of -16.
   (b) (5 points) Write $z^4 + 16$ as $(z - c_1)(z - c_2)(z - c_3)(z - c_4)$.
   (c) (5 points) Using that complex roots of a polynomial with real coefficients come in complex conjugate pairs, write $z^4 + 16$ as a product of two quadratic polynomial with real coefficients.

(a) $-16 = 16e^{\pi i}$ $\quad c = re^{i\theta}$, $c^4 = -16 \Leftrightarrow r^4 = 16$,

$4\theta = \pi + 2\pi n,\ n \in \mathbb{Z},\quad 0 \le \theta < 2\pi.$

Thus $r = 2$ & $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$,

$$c_1 = 2e^{\frac{\pi i}{4}} = \sqrt{2}(1+i),$$
$$c_2 = 2e^{\frac{3\pi i}{4}} = \sqrt{2}(-1+i),$$
$$c_3 = 2e^{\frac{5\pi i}{4}} = -\sqrt{2}(1+i),$$
$$c_4 = 2e^{\frac{7\pi i}{4}} = \sqrt{2}(1-i).$$

(b) & (c)

$$z^4 + 16 = (z - c_1)(z - c_2)(z - c_3)(z - c_4)$$

complex conjugate pairs

$$= (z - \sqrt{2} - i\sqrt{2})(z - \sqrt{2} + i\sqrt{2}) \cdot (z + \sqrt{2} - i\sqrt{2})(z + \sqrt{2} + i\sqrt{2})$$

$$= ((z - \sqrt{2})^2 + 2)((z + \sqrt{2})^2 + 2)$$

$$= (z^2 - 2\sqrt{2}z + 4)(z^2 + 2\sqrt{2}z + 4)$$

3. (a) (10 points) Find the image of the domain

$$\{z = x + iy \in \mathbb{C} : 0 < x^2 - y^2 < 1\}$$

under the mapping $w = z^2$.

(b) (15 points) Find the image of the infinite strip

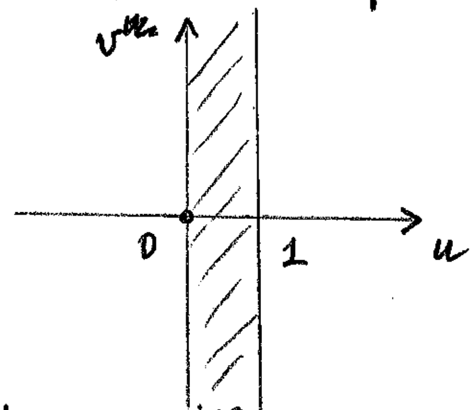
$$-\infty < x < \infty, \quad 0 \le y \le \frac{\pi}{4}$$

under the mapping $w = \exp z$, and label corresponding portions of the boundaries.

(a) $z = x + iy$, $w = z^2 = u + iv$, $u = x^2 - y^2$,
$v = 2xy$.

$0 < x^2 - y^2 < 1 \iff 0 < u < 1$

$\Rightarrow$ the image is vertical strip $0 < u < 1$ on (open)
uv-plane:

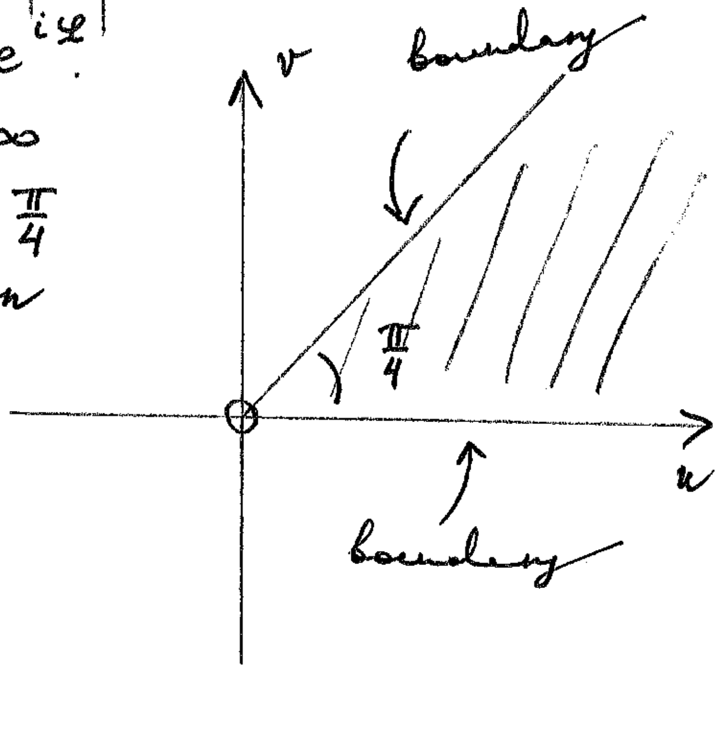


(b) $w = e^z = e^x e^{iy} = \rho e^{i\varphi}$.

$-\infty < x < \infty \iff 0 < \rho < \infty$

$0 \le y \le \frac{\pi}{4} \iff 0 \le \varphi \le \frac{\pi}{4}$

– The image is the sector on uv-plane bounded by two rays $\varphi = 0$ & $\varphi = \frac{\pi}{4}$, without the origin



4. (20 points) Suppose that $f(z)$ is continuous at z_0 and $f(z_0) = 0$. Using $\epsilon - \delta$ definition of a limit, show that

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = 0,$$

if there exist $a > 0$ and $M > 0$ such that $|g(z)| \geq M$ for all $|z - z_0| < a$.

f is continuous at z_0 & $f(z_0) = 0$, so $\forall \varepsilon > 0$ $\exists$ $\delta_1 > 0$ s.t. $|f(z)| < \varepsilon M$ whenever $|z - z_0| < \delta_1$. Put $\delta = \min\{a, \delta_1\}$. Then for $|z - z_0| < \delta$ we have $\dfrac{1}{|g(z)|} \leq \dfrac{1}{M}$

and

$$\left| \frac{f(z)}{g(z)} \right| = \frac{|f(z)|}{|g(z)|} < \frac{\varepsilon M}{M} = \varepsilon,$$

5. (15 points) Show that for every $z = x + iy$ the function

$$f(z) = \sinh x \cos y + i \cosh x \sin y$$

is differentiable and find $f'(z)$ as a function of z. What about the function $g(z) = \sinh x \cos y - i \cosh x \sin y$?

(a) $u = \sinh x \cos y$, $v = \cosh x \sin y$ have continuous partial derivatives everywhere, and

$u_x = \cosh x \cos y = v_y$
$u_y = -\sinh x \sin y = -v_x$ } CR equations are satisfied

$\Rightarrow f(z)$ is differentiable everywhere and

$f'(z) = u_x + iv_x = \cosh x \cos y + i \sinh x \sin y$

$= \frac{1}{2}(e^x + e^{-x})\cos y + \frac{i}{2}(e^x - e^{-x})\sin y$

1	2	3	4	5	TOTAL
20 points	20 points	25 points	20 points	15 points	100 points

$= \frac{1}{2}e^x(\cos y + i \sin y) + \frac{1}{2}e^{-x}(\cos y - i \sin y)$

$= \frac{1}{2}(e^z + e^{-z})$.

(b) Here v is $-v$ from part (a), so

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \Leftrightarrow \begin{cases} \cosh x \cos y = 0 \\ \sinh x \sin y = 0 \end{cases}$$

$\Leftrightarrow y = \frac{\pi}{2} + \pi n$, $n \in \mathbb{Z}$ & $x = 0$.

Thus $g(z)$ is differentiable only for $z = i(\frac{\pi}{2} + \pi n)$, $n \in \mathbb{Z}$, and for such z

$g'(z) = 0$.